A study of a simple gyrotron equation

2007 J. Phys. A: Math. Theor. 40 2203

(http://iopscience.iop.org/1751-8121/40/9/022)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 2203–2218 doi:10.1088/1751-8113/40/9/022

A study of a simple gyrotron equation

Markus Goetz[1], **Rita Meyer-Spasche**[1] **and Harold Weitzner**[2]

[1] Max-Planck-Institut für Plasmaphysik, EURATOM-Association, D-85748 Garching, Germany
[2] New York University, Courant Institute of Mathematical Sciences, New York, NY 10012, USA

Received 15 November 2006, in final form 8 January 2007
Published 14 February 2007
Online at stacks.iop.org/JPhysA/40/2203

Abstract
A simple standard equation for the evolution of the electrons and electromagnetic fields in a gyrotron cavity is studied. A number of mathematical properties are shown: existence and uniqueness of solutions for a limited axial extent and existence and uniqueness for all axial lengths in one case of particular interest. A Poynting theorem is obtained directly from the model and the Hamiltonian character of the electron motion is demonstrated. The start-up and final state in the gyrotron cavity are also examined. The efficiency of energy flux transfer from the electron beam to the wave is estimated.

PACS numbers: 52.35.Hr, 84.40.Ik

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Gyrotrons are electron tubes that currently provide both steady state and pulsed power in the frequency range of about 20–170 GHz and powers up to about 1 MW. Gyrotrons are also believed to be able to be developed in the future to produce very powerful electromagnetic radiation in a wider frequency range (optical, microwave, soft x-rays). In a gyrotron cavity the electron beam is guided by a longitudinal, constant magnetic field. The electrons undergo cyclotron oscillations and radiate at approximately that frequency. Although many more complex models are presently in use to represent gyrotron dynamics, there exist standard, relatively simple models that remain in active use. It is the purpose of this paper to explore some relevant mathematical and physical properties of one of these models (2, equations (2.60), (2.61)), (4, chapter 4). There is a vast literature on the derivation of these models. We cite only the relatively recent first monographs [4, 5]. They review the field and give more extensive references.

The model in question involves a paraxial approximation to the electron motion and to the fields. The transverse electron momentum density is $p(\zeta, \theta) = u(\zeta, \theta) + \mathrm{i}v(\zeta, \theta)$, where ζ is the axial coordinate and θ is a parameter which characterizes the spiralling position of the particle projected onto a plane $\zeta =$ constant. The parameter θ is scaled so that $0 \leqslant \theta \leqslant 2\pi$ and

$p(\zeta, \theta)$ is periodic in θ. The electromagnetic field is dominated by its x and y components, and the complex field is represented as $f(\zeta) = f_r(\zeta) + \mathrm{i} f_i(\zeta)$. It is also convenient to introduce $\frac{\mathrm{d}f}{\mathrm{d}\zeta} = \mathrm{i}g(\zeta)$. In appropriate dimensionless variables, $p(\zeta, \theta)$ satisfies the relativistic version of Newton's laws:

$$\frac{\mathrm{d}p(\zeta,\theta)}{\mathrm{d}\zeta} + \mathrm{i}\frac{1}{s}(c + |p|^2)p = \mathrm{i} f \bar{p}^{s-1}, \tag{1}$$

where $\bar{p}$ is the complex conjugate of p, s is the harmonic number of the oscillations and c is a real constant. We consider in detail the case $s = 1$ here, for which the fields are non-zero on axis, although we give many general results for all s. The electromagnetic field written as the second-order equation is

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\zeta^2} + \gamma(\zeta) f = \frac{d}{2\pi}\int_0^{2\pi} \mathrm{d}\theta [p(\zeta,\theta)]^s, \tag{2}$$

where d is a real constant and $\gamma(\zeta)$ models the variation of radius of the cavity as a function of ζ. Typically $\gamma(\zeta) < 0$ for $\zeta < 0$ and $\gamma(\zeta) > 0$ for ζ sufficiently large. When $\gamma(\zeta) < 0$, the electrons stimulate the production of microwaves. The integral on θ in (2) approximates the electron current driving the microwaves averaged over one gyro-oscillation. Provided the cavity parameters do not vary too rapidly, the average is a reasonable approximation. For convenience we rewrite (2) as a first-order system

$$\frac{\mathrm{d}f}{\mathrm{d}\zeta} = \mathrm{i}g(\zeta) \tag{3}$$

$$\frac{\mathrm{d}g}{\mathrm{d}\zeta} = \mathrm{i}\left\{\gamma(\zeta) f(\zeta) - \frac{d}{2\pi}\int_0^{2\pi} \mathrm{d}\theta [p(\zeta,\theta)]^s\right\}. \tag{4}$$

To complete the differential equations we must impose some initial conditions or two-point boundary conditions. We discuss this point further later in this paper, but we consider that in any case we expect to introduce an electron beam at $\zeta = 0$, so that the current distribution averaged over one gyration is

$$p(0, \theta) = p_s(\theta), \tag{5}$$

with $p_s(\theta)$ being a smooth, 2π-periodic function of θ, one typically assumes $p_s(\theta) = \mathrm{e}^{\mathrm{i}\theta}$. For much of the analysis we work with the straightforward initial value problem

$$f(0) = f_s, \tag{6}$$

$$g(0) = g_s. \tag{7}$$

We note in passing that this system is reversible in space in the sense that under the transformation $p \to \bar{p}, \zeta \to -\zeta, f \to \bar{f}, g \to \bar{g}, \gamma(\zeta) \to \gamma(-\zeta)$ the system is unchanged. Thus, whatever we show for $\zeta > 0$ holds equally for $\zeta < 0$.

In the next section, we present a number of general mathematical and physical properties of the system (1), (3), (4). The appearance of the parameter θ in (1) and (4) implies that the standard existence and uniqueness of solutions for the system is not immediately relevant. These proofs typically involve some iteration and limiting process. One must take care that in the limiting process one shows that the limiting function $p(\zeta, \theta)$ is at least smooth enough in θ that the integral in (2) or (4) exists and is the limit of the integral with the approximating functions. The proof is straightforward, but must be done. The existence and uniqueness is only for a limited interval in ζ, but applies for all values of s. When $s = 1$, special properties of the system allow one to show that solutions exist for all ζ and that they are as smooth in θ as the initial data. Thus, while $p(\zeta, \theta)$ may vary rapidly in θ, if it starts out as continuous or differentiable in θ then it remains of the same class. The existence for all ζ is of physical significance in that the phenomenon of beam bunching, which does certainly occur, cannot produce singularities or breakdown of solutions. Any numerical solutions which show breakdown are not computed with sufficient accuracy to represent the actual solution. The result may be true for other values of s, but our proof is not relevant in these cases. We then obtain directly from the system (1), (2) or (1), (3), (4) the Poynting theorem on the conservation of energy flux. Finally, we show that (1), where $f(\zeta)$ is considered an arbitrary function of ζ, is a Hamiltonian system for the particle motion. The result is not surprising, but again must be shown. Finally, we introduce the very convenient set of angle-action variables.

The third section considers only the case $s = 1$ and considers the idealization where $\gamma(\zeta) = -\Gamma_1$ for $z_1 \leqslant \zeta \leqslant 0$, and $\gamma(\zeta) = +\Gamma_2$ for $\zeta \geqslant z_2$, where Γ_1 and Γ_2 are positive. A linearized perturbation of the system in $z_1 \leqslant \zeta \leqslant 0$, linearized about the state $p = \mathrm{e}^{\mathrm{i}\theta - \mathrm{i}(c+1)\zeta}$, $f = g = 0$, enables one to determine how a gyrotron starts up. We show there is essentially a unique start-up mode, so that with f_s and g_s chosen consistent with this mode the initial value problem is reasonable. We then turn to the final state and ask under what conditions is a solution to the system with $f(\zeta) = F\mathrm{e}^{-\mathrm{i}k\zeta}$ possible? We determine all such solutions which are dynamically accessible from the initial state. We give examples of such final states for the case of $|F|$ small. With such a choice, perturbation analysis allows a fairly complete characterization of the possible states. In some cases only one final state is possible, while in other cases multiple final states are possible. We are also able to determine the efficiency of the energy flux conversion from the beam to the wave. We note that in many cases of physical relevance, see e.g. reference [2], F is indeed small.

The last section of this paper summarizes the results.

2. Some general properties of the model

2.1. Existence and uniqueness of solutions in ζ and θ

The existence and uniqueness proof for the system (1), (3), (4) requires a relatively straightforward extension of the usual Picard–Lindelöf iteration argument. To show the solution exists in some finite time interval, one first shows that one can carry out an iteration with uniform bounds on the iterates. One can then easily show that the iterates form an equicontinuous set of functions in the parameter θ. The usual arguments then show that the iterates converge, and equicontinuity together with the standard arguments show that the iterates converge to a solution. The uniqueness is exactly the same as in the standard case. One can see this program carried out in the thesis [2] for the special case $s = 1$.

We now sketch in more detail the elements of the proof of the theorem that there is a positive constant C such that in $0 \leqslant \zeta < C$ the problem possesses exactly one solution within the class of functions continuous in ζ and θ and with $p(\zeta, \theta)$ periodic of period 2π in θ. We state a series of elementary lemmas which lead to the proof of the theorem.

Let B be a function space the elements S of which represent a triple of functions $S := (p(\zeta, \theta), f(\zeta), g(\zeta))$ continuous in their arguments in $0 \leqslant \zeta < C_1, 0 \leqslant \theta \leqslant 2\pi$, and periodic of period 2π in θ. Let $\|S\| \equiv \sup_\theta |p(\zeta, \theta)| + |f(\zeta)| + |g(\zeta)|$ and let Σ be the set of all such functions with norm less or equal to $2\left(\sup_\theta |p_s(\theta)| + |f_s| + |g_s|\right)$.

We rewrite the system of differential equations as an integral equation by means of the operator O, where

$$OS = \left(p_s(\theta) + \mathrm{i}\int_0^\zeta \left\{f(\zeta')[\bar{p}(\zeta', \theta)]^{s-1} - \frac{1}{s}(c + |p|^2)p(\zeta')\right\} \mathrm{d}\zeta',\right.$$

$$f_s + \mathrm{i}\int_0^\zeta g(\zeta')\,\mathrm{d}\zeta',$$

$$g_s + \mathrm{i}\int_0^\zeta \left\{\gamma(\zeta')f(\zeta') - \frac{d}{2\pi}\int_0^{2\pi}[p(\zeta',\theta)]^s\,\mathrm{d}\theta\right\}\mathrm{d}\zeta'\Bigg). \tag{8}$$

Clearly, any fixed point of O within the function space B is a solution of the differential equations.

Lemma 1. *There is a $C_2 > 0$ such that for $0 \leqslant \zeta < C_2$ the set Σ is closed under the operation O.*

Proof. Obvious as the integrands are uniformly bounded in Σ. □

Lemma 2. *There is a $K_1 > 0$ such that for $S_1(\zeta,\theta)$ and $S_2(\zeta,\theta)$ in Σ and $0 \leqslant \zeta < C_2$*

$$\|OS_1(\zeta,\theta) - OS_2(\zeta,\theta)\| \leqslant K_1\int_0^\zeta \|S_1(\zeta',\theta) - S_2(\zeta',\theta)\|\,\mathrm{d}\zeta'. \tag{9}$$

Proof. Obvious as the integrands can be written as sums of products of a uniformly bounded function times $\|S_1(\zeta') - S_2(\zeta')\|$. □

Corollary

$$\sup_{0\leqslant\zeta'\leqslant\zeta}\|OS_1(\zeta',\theta) - OS_2(\zeta',\theta)\| \leqslant K\zeta \sup_{0\leqslant\zeta'\leqslant\zeta}\|S_1(\zeta',\theta) - S_2(\zeta',\theta)\|.$$

Corollary. *There is a $C_3 > 0$ such that for $0 \leqslant \zeta < C_3$ O is a contractive mapping on Σ in the norm*

$$|||S(\zeta,\theta)||| = \sup_{0\leqslant\zeta'\leqslant\zeta}\|S(\zeta',\theta)\|.$$

Lemma 3. *Let S_1 be any element of Σ, and let $p_1(\zeta,\theta)$ be the corresponding component of S_1 and $Op_1(\zeta,\theta)$ the corresponding component of OS_1, then there is a constant K_2*

$$|Op_1(\zeta,\theta) - Op_1(\zeta,\theta')| \leqslant |p_s(\theta) - p_s(\theta')| + K_2\int_0^\zeta |p_1(\zeta',\theta) - p_1(\zeta',\theta')|\,\mathrm{d}\zeta'. \tag{10}$$

Proof. Obvious from the structure of the integrands and the uniform boundedness of the various functions. □

Lemma 4. *Let S_n be the sequence of functions O^nS_0, where S_0 is an arbitrary element of Σ, then there is a $C_4 > 0$ such that for $0 \leqslant \zeta < C_4$ the sequence $\{S_n\}$ converges uniformly to a unique element of Σ.*

Proof. Pick $C_4 < \inf(C_3, 1/(K_2+1))$, then for $0 \leqslant \zeta < C_4$

$$\sup_\zeta |Op_1(\zeta,\theta) - Op_1(\zeta,\theta')| \leqslant |p_s(\theta) - p_s(\theta')| + \eta\sup_\zeta|p_1(\zeta,\theta) - p_1(\zeta,\theta')|, \tag{11}$$

where $\eta < 1$. Hence, $\{S_n\}$ converges uniformly to a limiting function which is clearly continuous in ζ. Let $p_n(\zeta,\theta)$ be the corresponding element of S_n. The estimate (11) shows that

$$\lim_{n\to\infty}\sup_\zeta|p_n(\zeta,\theta) - p_n(\zeta,\theta')| \leqslant \frac{|p_s(\theta) - p_s(\theta')|}{1-\eta}. \tag{12}$$

Thus, the limiting function is also continuous in θ and periodic of period 2π. The uniqueness of the limiting function is also immediate from (9). □

Lemma 5. *There is at most one fixed point in* Σ *for* $0 \leqslant \zeta < C_4$.

Proof. Obvious from contractive mapping.

Hence, the theorem is proved. □

2.2. *Existence and uniqueness for all* ζ *for* $s = 1$

Theorem. *The problem possesses a (unique) solution for all* ζ *when* $s = 1$.

Lemma 1. *Let the problem have a solution* $S(\zeta, \theta)$ *in the interval* $0 \leqslant \zeta < D$*, then there is a constant* L *such that* $\|S(\zeta, \theta)\| \leqslant \|S(0, \theta)\| e^{L\zeta}$.

Proof. If we introduce the real function $\psi(\zeta, \theta)$ where

$$\psi(\zeta, \theta) = \int_0^\zeta [c + |p(\zeta', \theta)|^2] \, d\zeta' \tag{13}$$

then from (1)

$$p(\zeta, \theta) = e^{i\psi(\zeta,\theta)} \left[p_s(\theta') + i \int_0^\zeta f(\zeta') \, e^{i\psi(\zeta',\theta)} \, d\zeta' \right], \tag{14}$$

$$\sup_{(\theta)} |p(\zeta, \theta)| \leqslant \sup_\theta |p_s(\theta)| + \int_0^\zeta |f(\zeta')| \, d\zeta'. \tag{15}$$

From (8) we also have

$$|f(\zeta)| \leqslant |f_s| + \int_0^\zeta |g(\zeta')| \, d\zeta' \tag{16}$$

$$|g(\zeta)| \leqslant |g_s| + \sup_\zeta |\gamma(\zeta)| \int_0^\zeta |f(\zeta')| \, d\zeta' + \frac{|d|}{2\pi} \int_0^\zeta d\zeta' \sup_\theta |p(\zeta', \theta)|. \tag{17}$$

Hence,

$$\|S(\zeta, \theta)\| \leqslant \|S(0, \theta)\| + (1 + |\Gamma| + |d|) \int_0^\zeta \|S(\zeta', \theta)\| \, d\zeta' \tag{18}$$

where

$$\Gamma = \sup_\zeta |\gamma(\zeta)|. \tag{19}$$

The result now follows from the Grönwall inequality. □

Lemma 2. $S(\zeta, \theta)$ *is uniformly continuous in* θ *for all* ζ *in the interval* $0 \leqslant \zeta < D$.

Proof. We return to (10). Since the preceding lemma implies that S is uniformly bounded in $0 \leqslant \zeta < D$, the constant K_2 exists for the entire interval $0 \leqslant \zeta < D$. Let $p^{(n)}(\zeta, \theta)$ be the nth approximation to $p_1(\zeta, \theta)$, where $p^{(0)}(0, \theta) = p_s(\theta)$. Then it follows easily from (10) by iteration that

$$|p^{(n)}(\zeta,\theta) - p^{(n)}(\zeta,\theta')| \leqslant e^{K_2\zeta}|p_s(\theta) - p_s(\theta')|. \tag{20}$$

The uniformity of continuity follows. □

Lemma 3. *The solution $S(\zeta,\theta)$ can be extended to $\zeta = D$ and is continuous in θ for $\zeta = D$. The existence of uniformly bounded derivative $\frac{dS}{d\zeta}(\zeta,\theta)$ implies that $S(\zeta,\theta)$ can be extended continuously to $\zeta = D$, and (20) shows that $S(\zeta,\theta)$ is continuous in θ.*

Corollary. *The solution can be extended for some interval with $\zeta > D$.*

Proof. Apply the existence and uniqueness theorem of the previous section.

Hence, it is impossible that there exist a constant E such that a solution exists for $0 \leqslant \zeta < E$ but for no larger interval.

Thus, the theorem is proved. □

2.3. *Energy flux conservation*

We see from (1) and its complex conjugate that

$$d\left\{\bar{p}\frac{dp}{d\zeta} + p\frac{d\bar{p}}{d\zeta}\right\} = id(f\bar{p}^s - \bar{f}p^s), \tag{21}$$

while from (4)

$$\bar{f}\frac{dg}{d\zeta} + f\frac{d\bar{g}}{d\zeta} = i\frac{d}{2\pi}\left\{-\bar{f}\int_0^{2\pi} d\theta[p(\zeta,\theta)]^s + f\int_0^{\pi} d\theta[\bar{p}(\zeta,\theta)]^s\right\}, \tag{22}$$

so that

$$\frac{d}{d\zeta}\left\{\frac{d}{2\pi}\int_0^{2\pi}\bar{p}p\,d\theta - \bar{f}g - f\bar{g}\right\} = 0 \tag{23}$$

or

$$\frac{d}{2\pi}\int_0^{2\pi}|p|^2\,d\theta + 2\,\mathrm{Im}\,\bar{f}'f = \text{constant}. \tag{24}$$

This result is the energy flux conservation, or Poynting, theorem.

2.4. *Hamiltonian structure of the electron motion*

If we write $p(\zeta,\theta) = u(\zeta,\theta) + iv(\zeta,\theta)$, then the equation of motion for the electron becomes

$$\frac{du}{d\zeta} = \frac{1}{s}(c + (u^2 + v^2))v - \mathrm{Im}[f(\zeta)(u - iv)^{s-1}] \tag{25}$$

$$\frac{dv}{d\zeta} = -\frac{1}{s}(c + u^2 + v^2)u + \mathrm{Re}[f(\zeta)(u - iv)^{s-1}]. \tag{26}$$

If we set

$$H(u,v;\zeta) := \frac{1}{s}c\frac{u^2+v^2}{2} + \frac{1}{s}\frac{(u^2+v^2)^2}{4} - \mathrm{Re}\,f(\zeta)\frac{(u-iv)^s}{s} \tag{27}$$

then

$$\frac{du}{d\zeta} = \frac{\partial H}{\partial v} \quad \text{and} \quad \frac{dv}{d\zeta} = -\frac{\partial H}{\partial u}. \tag{28}$$

An immediate and important consequence of the Hamiltonian character of *autonomous systems* is that the Hamiltonian function is constant on trajectories and that the system is area preserving.

In (25)–(28), however, we assume that f is an arbitrary smooth function of ζ. Because of this ζ-dependence, H is not autonomous and thus not constant on trajectories. But H generates an area-preserving transformation of the phase plane [3]. Specifically, let $\mathcal{D}(u, v, \zeta)$ be a connected domain in the (u, v) plane, which depends on ζ. Each point in $\mathcal{D}$ changes with respect to ζ according to the equations of motion (25), (26) or, equivalently, (27), (28). Then, the area of $\mathcal{D}$ does not change with ζ or

$$\int_{\mathcal{D}(u,v,\zeta)} \mathrm{d}u\,\mathrm{d}v = \text{constant}. \tag{29}$$

For the initial data (5) the area inside the curve $p(0, \theta) = \mathrm{e}^{\mathrm{i}\theta}$ is just π, so that any state accessible from the initial state (5) must also have an area π. Similarly since this initial state with $f_s \sim g_s \sim 0$ has energy flux d, any state accessible from this state must also have energy flux d.

We conclude this section with a possible standard change of variables from (u, v) to angle-action variables. If we set

$$u = \sqrt{2I}\cos\psi \tag{30}$$

$$v = \sqrt{2I}\sin\psi, \tag{31}$$

then

$$\frac{\partial(I, \psi)}{\partial(u, v)} = 1, \tag{32}$$

so that the transformation is ζ-independent and area preserving.

From these properties, or directly from the differential equations (25), (26), one concludes that if one inserts (30), (31) into (27) then

$$H(I, \psi) = cI/s + I^2/s - \mathrm{Re}((2I)^{s/2}\,\mathrm{e}^{-\mathrm{i}s\psi} f)/s \tag{33}$$

and the equations of motion for I and ψ are Hamilton's equations with Hamiltonian (33). If $f(\zeta)$ is in fact independent of ζ then the solutions of the equations of motion are $H =$ constant, where H is given by (27) or (33). We shall use these results in the next section.

3. Initial and final states in a gyrotron

For this section, we examine the nature of plausible initial and final states in a gyrotron. We idealize the system and assume that for $\zeta \leqslant 0, \gamma(\zeta) = -\Gamma_1$, and for $\zeta \geqslant z_2, \gamma(\zeta) = +\Gamma_2$, where Γ_1 and Γ_2 are positive. We recognize that this is not physically practicable, but it enables us to explore the nature of the solutions and describe a class of accessible final states. Although we have written the following subsections for the case = 1 all the results can be easily extended to arbitrary s.

3.1. Initial states

We start with the simpler case of the initial states or gyrotron start-up.

It is clear that an exact solution of our system (1), (2) with $p(0, \theta) = p_s(\theta)$ is

$$p(\zeta, \theta) = p_s(\theta)\exp[-\mathrm{i}\zeta(c + |p_s(\theta)|^2)], \tag{34}$$

$$f = 0. \tag{35}$$

We put $p_s(\theta) = \exp\mathrm{i}\theta$, linearize about this state and write

$$p = \exp \mathrm{i}[\theta - \zeta(c+1)]\{1 + p_1(\zeta, \theta) + \cdots\} \tag{36}$$

$$f = f_1(\zeta)\, \mathrm{e}^{-\mathrm{i}\zeta(c+1)} + \cdots \tag{37}$$

then

$$p_{1,\zeta} + \mathrm{i}(p_1 + \bar{p}_1) = \mathrm{i}\, f\, \mathrm{e}^{\mathrm{i}\theta} \tag{38}$$

$$f_{1,\zeta\zeta} - 2\mathrm{i}(c+1) f_{,\zeta} - [\Gamma_1 + |c+1|^2] f_1 = \frac{d}{2\pi} \oint \mathrm{d}\theta\, p_1(\zeta, \theta)\, \mathrm{e}^{+\mathrm{i}\theta}. \tag{39}$$

Before we apply a Laplace transform it is useful to separate real and imaginary parts

$$p_1 = u + \mathrm{i}v, \qquad f = f_r + \mathrm{i} f_i, \tag{40}$$

so that

$$u_{,\zeta} = (f_r \sin\theta - f_i \cos\theta) \tag{41}$$

$$v_{,\zeta} + 2u = (f_r \cos\theta + f_i \sin\theta). \tag{42}$$

When we assume that the dependent variables vary as $\mathrm{e}^{\sigma\zeta}$, $\mathrm{Re}\, \sigma > 0$, we find

$$u = (f_r \sin\theta - f_i \cos\theta)/\sigma \tag{43}$$

$$v = -2(f_r \sin\theta - f_i \cos\theta)/\sigma^2 + (f_r \cos\theta + f_i \sin\theta)/\sigma. \tag{44}$$

It is now easy to evaluate

$$\frac{1}{2\pi} \int p_1\, \mathrm{e}^{\mathrm{i}\theta} \mathrm{d}\theta = \frac{1}{2\pi} \int (u \cos\theta - v \sin\theta)\, \mathrm{d}\theta + \frac{\mathrm{i}}{2\pi} \int (u \sin\theta + v \cos\theta)\, \mathrm{d}\theta \tag{45}$$

$$= f \left(\frac{1}{\sigma^2} + \frac{\mathrm{i}}{\sigma} \right), \tag{46}$$

so that

$$\sigma^2 [\sigma - \mathrm{i}(c+1)]^2 - \Gamma_1 \sigma^2 = d(1 + \mathrm{i}\sigma). \tag{47}$$

If we set $\sigma = \mathrm{i}\tau$, then the equation for τ is

$$\tau^2 [\Gamma_1 + (\tau - c - 1)^2] - d(1 - \tau) = 0. \tag{48}$$

Since the polynomial is negative for $\tau = 0$ and positive for sufficiently large values of $|\tau|$, there are always at least two real roots, one positive and one negative. These roots correspond to purely oscillating solutions of the system. When $0 < d \ll 1$, there are also two other modes of the system

$$\sigma = \mathrm{i}(c+1) \pm \sqrt{\Gamma_1} + 0(d). \tag{49}$$

Clearly, the mode with the plus sign is growing as ζ increases and tends to zero as ζ tends to $-\infty$. Thus, reasonable starting values for the differential equations would be to initialize this mode with the plus sign in (49). The values would then be

$$f_1 = f_s \tag{50}$$

$$g_1 = -\mathrm{i}\sqrt{\Gamma_1} f_1 \tag{51}$$

and the modifications to p are given by (36), (43) and (44). The actual value of f_s is arbitrary, but it should be small. The modifications to p should also be included in the initialization. It would not be reasonable to mix in any of the purely oscillating solutions, as they cannot grow out of noise, as the chosen solution can. The specification of initial data for f_1, g_1 and p_1 converts the two-point boundary value problem to an initial value problem. For the reasons just given, the change in the problem seems appropriate.

3.2. Final states

We now turn to the final state of the gyrotron for $s = 1$. The desirable final state should be a pure outgoing wave, that is $f = F\,\mathrm{e}^{-\mathrm{i}k\zeta}$, where $k > 0$, and without loss of generality F is a real constant. We turn to the Poynting theorem (24), evaluate the constant in the initial state with $f = 0$ and $p = \mathrm{e}^{\mathrm{i}\theta}$ and conclude for $\zeta \geqslant z_2$ that

$$2kF^2 = d\left\{1 - \frac{1}{2\pi}\int_0^{2\pi} |p(\zeta,\theta)|^2\,\mathrm{d}\theta\right\}. \tag{52}$$

The wave equation (2) implies

$$(\Gamma_2 - k^2)F = \frac{d\,\mathrm{e}^{\mathrm{i}k\zeta}}{2\pi}\int_0^{2\pi} p(\zeta,\theta)\,\mathrm{d}\theta. \tag{53}$$

Thus, (52) and (53) are two constraints that the final state must satisfy. We have one additional constraint, the constancy of the area inside the curve in the phase plane $p(\zeta,\theta), 0 \leqslant \theta \leqslant 2\pi$ or (29). We defer giving this constraint explicitly until we give the final state more precisely.

If we introduce $\tilde{p}(\zeta,\theta)$ as

$$p(\zeta,\theta) = \tilde{p}(\zeta,\theta)\,\mathrm{e}^{-\mathrm{i}k\zeta} \tag{54}$$

then $\tilde{p}(\zeta,\theta)$ satisfies the equation

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\zeta} + \mathrm{i}(c - k + |\tilde{p}|^2)\tilde{p} = \mathrm{i}F \tag{55}$$

and (52), (53) becomes

$$2kF^2 = d\left\{1 - \frac{1}{2\pi}\int_0^{2\pi} |\tilde{p}(\zeta,\theta)|^2\,\mathrm{d}\theta\right\} \tag{56}$$

$$(\Gamma_2 - k^2)F = \frac{d}{2\pi}\int_0^{2\pi} \tilde{p}(\zeta,\theta)\,\mathrm{d}\theta. \tag{57}$$

We see that (55) has exactly the same structure as the original equation (1), except that the inhomogeneous term is independent of ζ. Thus, (55) is a Hamiltonian system with Hamiltonian given by (27) or (33), where c is replaced by $c - k$ and $f(\zeta)$ is replaced by the real constant F. In particular, the solution of (55) for θ fixed is exactly the curve $H =$ constant passing through the initial point. In principle, one could take any closed curve $p = p_f(\theta), 0 \leqslant \theta \leqslant 2\pi$ which encloses an area of π and construct the solution $p(\zeta,\theta)$, where $p(\zeta_0,\theta) = p_f(\theta)$, and allow each point on the curve to move according to Hamilton's equations or $H =$ constant. It would seem unlikely that such a solution could also have the right-hand side of (56), (57) independent of ζ, as required.

We show that there is a simple method to construct solutions $\tilde{p}(\zeta,\theta)$ of (58) which enclose an area π and which also can satisfy (56) and (57). We cannot guarantee that we have all such possible solutions, but it appears likely. To describe the construction of the solutions we turn to the dynamical system (55), with Hamiltonian (27) or (33) and replacement $c \to c - k$ and $f(\zeta) \to F$. If we introduce $\tilde{u}$ and $\tilde{v}$ as the real and imaginary parts of $\tilde{p}$, and the angle-action variables $\tilde{I} = \frac{1}{2}(\tilde{u}^2 + \tilde{v}^2)$, $\tilde{\psi} = \tan^{-1}\tilde{v}/\tilde{u}$, then the Hamiltonian is

$$H = (c - k)\left(\frac{\tilde{u}^2 + \tilde{v}^2}{2}\right) + \frac{(\tilde{u}^2 + \tilde{v}^2)^2}{4} - F\tilde{u}, \tag{58}$$

$$= (c - k)\tilde{I} + \tilde{I}^2 - F\sqrt{2\tilde{I}}\cos\tilde{\psi}. \tag{59}$$

3.2.1. Critical points of the Hamiltonian. The critical points of the Hamiltonian at which $\frac{\partial H}{\partial \tilde{u}} = \frac{\partial H}{\partial \tilde{v}} = 0$ satisfy

$$(c - k + \tilde{u}^2 + \tilde{v}^2)\tilde{v} = 0, \qquad (c - k + \tilde{u}^2 + \tilde{v}^2)\tilde{u} - F = 0. \tag{60}$$

Remark. Note that (58) is symmetric w.r.t. $\tilde{v} = 0$ and that it does not change if we replace both F by $-F$ and $\tilde{u}$ by $-\tilde{u}$. Thus, the critical points of the Hamiltonian are placed symmetrically about the $\tilde{u}$-axis. The replacement of F by $-F$ leads to the reflection of the curves $H =$ constant in the $\tilde{v}$-axis.

For $F = 0$ and $c - k > 0$, there is only one critical point, the origin. It is an O point. We now assume $F = 0$ and $c - k < 0$, i.e. $k - c > 0$. In this case, there is the critical point $(\tilde{u}, \tilde{v}) = (0, 0)$ and a circle of critical points with radius $\sqrt{k - c}$ and centre $(0, 0)$. The related dynamics is most easily analysed via the adapted ($c \to c - k$, $p_s(\theta) \to \tilde{p}_s(\theta)$) exact solution (34) (see also [1], where equation (1) was analysed for prescribed rhs, i.e. for the cold-cavity gyrotron model).

If $0 < |\tilde{p}_s(\theta)| < \sqrt{k - c}$, then the trajectories are circles around the origin with radius $|\tilde{p}_s|$. The phase point rotates counterclockwise with period $2\pi(k - c - |\tilde{p}_s|^2)^{-1}$.

If $|\tilde{p}_s| > \sqrt{k - c}$, then the trajectories are also circles around the origin with radius $|\tilde{p}_s|$, but the phase point rotates clockwise, with period $2\pi(|\tilde{p}_s|^2 - (k - c))^{-1}$.

Note that the period of the phase point tends to infinity if $|\tilde{p}_s|$ approaches $\sqrt{k - c}$ from below or from above.

We now turn to the case $F \neq 0$. Without restriction we assume $F > 0$. In this case, equation (60) reduces to

$$\tilde{v} = 0, \qquad (c - k)\tilde{u} + \tilde{u}^3 - F = 0.$$

If $c - k > 0$, then there is exactly one critical point, a centre, and all orbits are simple closed curves, see figure 1. Here and in the cases following each trajectory $\tilde{p}(\zeta) = \tilde{u} + \mathrm{i}\tilde{v}$ is closed, and thus represents a solution of Hamilton's equations with some period, which we denote as τ, so that

$$\tilde{p}(\zeta + \tau) = \tilde{p}(\zeta). \tag{61}$$

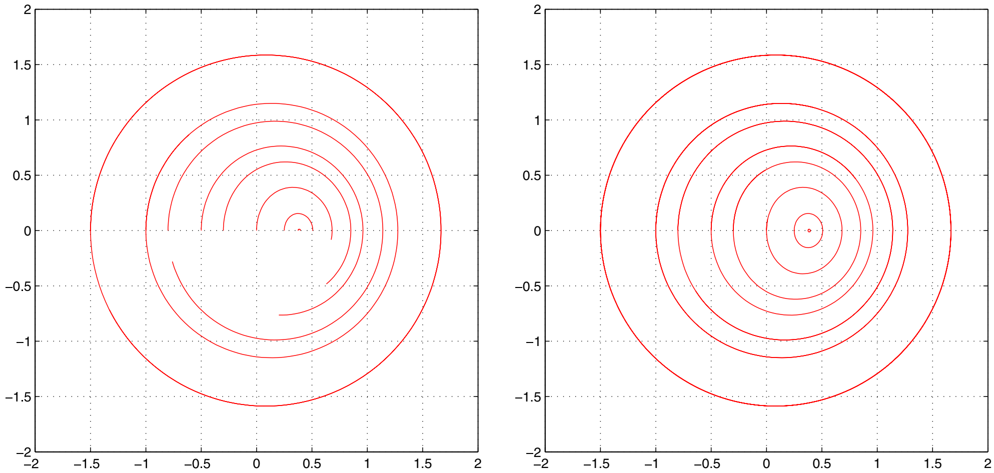
Computations were done in MATLAB with `ode45` for $0 \leqslant \zeta \leqslant \zeta_f$. We used the same ζ_f for all orbits in the same subfigure, but different ζ_f for the different subfigures because the values of τ are very different for different parameter values and different trajectories. Figures 1(*a*) and later 3(*a*) also show graphically how much of an orbit is traversed in a distance $0 \leqslant \zeta \leqslant \zeta_f$.

If $c - k < 0$, there are either one or three critical points. The transition occurs at $F = F_d$,

$$F_d = 2\left(\frac{k - c}{3}\right)^{3/2},$$

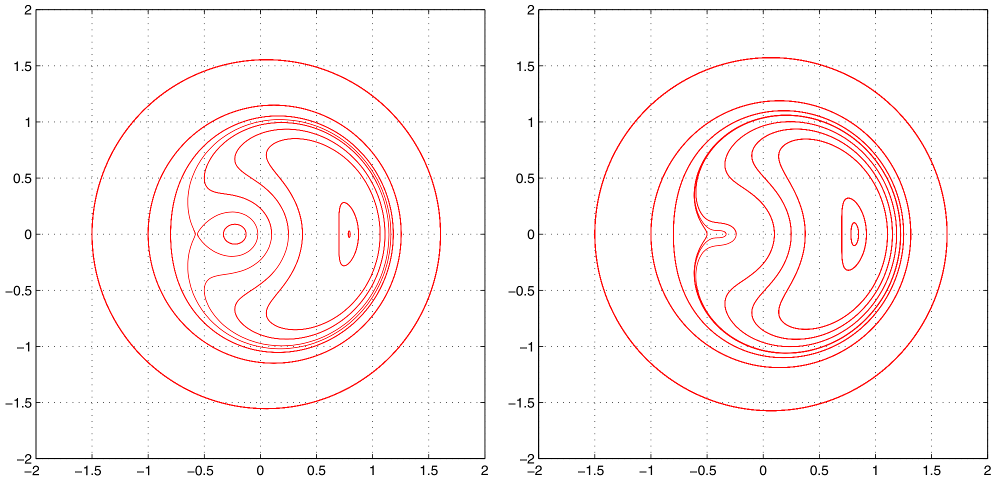
where two of the three critical $\tilde{u}$-values coincide and turn complex conjugate. For all values of F, the real parts of all three critical $\tilde{u}$-values add up to zero.

If $c - k < 0$ and $0 < F < F_d$, then there are exactly one X-point and two O-points. Again all orbits are closed, see figure 2. If $c - k < 0$ and $F = F_d$, the X-point and one of the O-points coincide. If $c - k < 0$ and $F > F_d$, there is only one centre and all orbits are closed, see figure 3.



(*a*) (*b*)

Figure 1. Computed orbits $(\tilde{u}(\zeta), \tilde{v}(\zeta)), 0 \leqslant \zeta \leqslant \zeta_f$, for the Hamiltonian (58) with parameter values $k - c = -0.5 < 0$ and $F = 0.25$. All initial values are of type $(\tilde{u}_0, 0)$ with $\tilde{u}_0 \leqslant 0.375$, the approximate value at the centre. (*a*) $\zeta_f = 4 < \tau$ for most orbits shown; (*b*) $\zeta_f = 8 > \tau$ for all orbits shown.



(*a*) (*b*)

Figure 2. Computed orbits $(\tilde{u}(\zeta), \tilde{v}(\zeta)), 0 \leqslant \zeta \leqslant \zeta_f$, for the Hamiltonian (58) with $k - c = 0.5 > 0$. Values of ζ_f such that $\zeta_f > \tau$ for all orbits shown. Those trajectories close to the *X*-points have the largest τ. (*a*) $F = 0.1 < F_d, \zeta_f = 40$; (*b*) $F = 0.136 \approx F_d, \zeta_f = 60$.

3.2.2. Solutions satisfying the constraints. We can now construct a solution of the system $\tilde{p}(\zeta, \theta)$ which can reasonably be expected to satisfy the area condition and also the conditions (56) and (57). We assume that for all values of ζ and θ the solution $\tilde{p}(\zeta, \theta)$ lies on one closed periodic trajectory, and we identify this trajectory as $\tilde{q}(\zeta)$ with period τ. We must now assign θ values on each point of the trajectory. We set

$$\tilde{p}(\zeta, \theta) = \tilde{q}(\zeta + \tau\theta/2\pi). \tag{62}$$

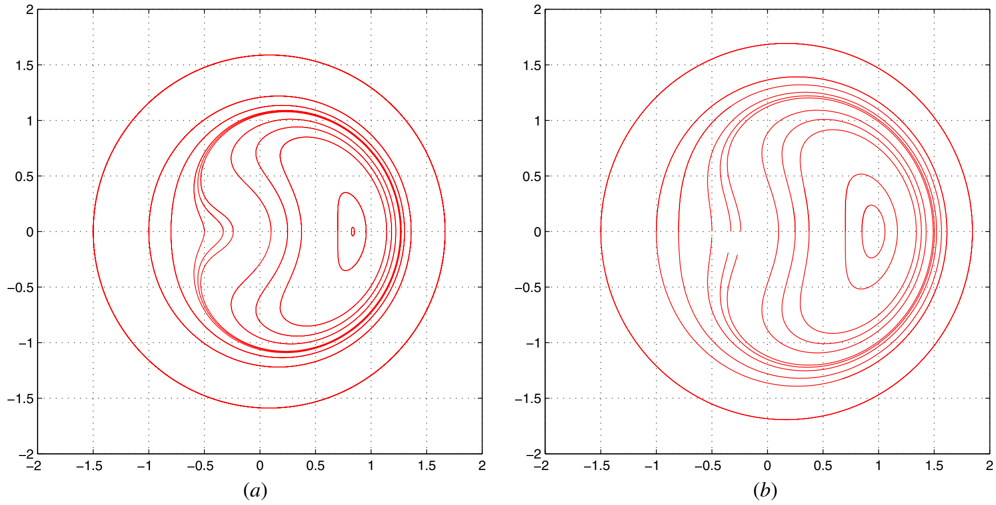


Figure 3. Computed orbits $(\tilde{u}(\zeta), \tilde{v}(\zeta)), 0 \leqslant \zeta \leqslant \zeta_f$, for the Hamiltonian (58) with $k - c = 0.5 > 0$. Values of ζ_f such that $\zeta_f > \tau$ for most orbits shown: in part (*b*), $\zeta_f = 8 < \tau < 9$ for the non-closed orbits shown. (*a*) $F = 0.167 > F_d, \zeta_f = 30$; (*b*) $F = 0.4, \zeta_f = 8$.

Clearly, this function satisfies Hamilton's equations, is periodic of period τ in ζ and periodic of period 2π in θ. We comment on other possibilities after we show that this choice allows (56) and (57) to be satisfied. We observe that for any smooth function $G(\zeta + \tau\theta/2\pi)$, which is periodic of period τ in ζ,

$$\frac{1}{2\pi}\int_0^{2\pi} G(\zeta + \tau\theta/2\pi)\,\mathrm{d}\theta = \frac{1}{\tau}\int_0^{\tau} G(\zeta)\,\mathrm{d}\zeta = \frac{\int_0^\tau G(\zeta)\,\mathrm{d}\zeta}{\int_0^\tau \mathrm{d}\zeta}. \tag{63}$$

If we represent the trajectory in angle-action variables, then

$$\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} = -\frac{\partial H}{\partial I} = -(c-k) - 2I + F(2I)^{-1/2}\cos\psi \tag{64}$$

and $\tilde{I} = \tilde{I}(\tilde{\psi})$ so that (56) and (57) become

$$\frac{2kF^2}{d} + \frac{2\int_0^{2\pi} I(\psi)\,\mathrm{d}\psi\left(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)^{-1}}{\int_0^{2\pi}\mathrm{d}\psi\left(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)^{-1}} = 1 \tag{65}$$

$$(\Gamma_2 - k^2)F = d\,\frac{\int_0^{2\pi}\tilde{q}(\tilde{\psi})\,\mathrm{d}\tilde{\psi}\left(\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\zeta}\right)^{-1}}{\int_0^{2\pi}\mathrm{d}\tilde{\psi}\left(\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\zeta}\right)^{-1}}. \tag{66}$$

Finally, the area condition expressed in terms of angle-action variable is

$$\frac{1}{\pi}\int_0^{2\pi}\tilde{I}(\tilde{\psi})\,\mathrm{d}\tilde{\psi} = 1. \tag{67}$$

The three conditions (65)–(67) are manifestly independent of ζ.

The integrals in (63), (65) and (66) assume that the orbit is periodic of period 2π in ψ. In some cases the orbit does not cover this range in ψ and is double-valued in ψ. In these cases the integrals are to be understood as an integral over the closed orbit, see Case III, examined in subsection (3.2.4).

3.2.3. Other choices Before we explore the consequences of (64)–(66) further we comment on other choices instead of (62). It seems reasonable that $\tilde{p}(\zeta, \theta)$ should lie on one trajectory of $H =$ constant. The question is how to assign θ values to points on the trajectory. We require that the orbit must be periodic in ζ of period τ and periodic in θ of period 2π. Suppose we define

$$\theta = \phi + \Phi(\phi), \tag{68}$$

where $\Phi(\phi)$ is periodic of period 2π in ϕ and $\Phi'(\phi) > -1$, so that $\phi + \Phi(\phi)$ is monotone. In that case another possible function $\tilde{p}(\zeta, \theta)$ would be

$$\tilde{p}(\zeta, \theta) = \tilde{q}(\zeta + \tau\phi(\theta)/2\pi). \tag{69}$$

However, the expression analogous to (63) would be

$$\frac{1}{2\pi}\int_0^{2\pi} G(\zeta + \tau\phi(\theta)/2\pi)\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} G(\zeta + \tau\phi/2\pi)(1 + \Phi'(\phi))\,\mathrm{d}\phi \tag{70}$$

and there is little reason to suspect that one could select $\Phi(\phi)$ so that the expressions would be independent of ζ. Thus, while we cannot be assured that (62) is the only possibility, it is a reasonable expectation.

We note that in terms of the original dependent variable

$$p(\zeta, \theta) = \tilde{q}(\zeta + \tau\theta/2\pi)\,\mathrm{e}^{-\mathrm{i}k\zeta}. \tag{71}$$

Thus, except for the special cases in which τ and $2\pi/k$ are commensurable, the particle motion defined by (71) is not periodic, it is only almost periodic. If one wishes to explore any possible periodicity one must consider the combination $p(\zeta, \theta)\,\mathrm{e}^{\mathrm{i}k\zeta}$. We note also that although we have written (62)–(67) for the case $s = 1$ all the expressions apply equally well for arbitrary s if one only replaces (64) by the corresponding expression obtained from (33).

3.2.4. Consequences. In order to explore the consequences of our analysis, we present a few perturbation expansion solutions of our system. Although we do not consider the matter in detail, the expansions are convergent in the parameters we choose. We expand the orbits under the assumption that $|F|$ is small. We use the angle-action variables to represent the orbits and omit $\sim$ from the variables, which are now q, I and ψ.

In order to compare the properties of the different classes of solutions, it is necessary to introduce a figure of merit for the different states. The Poynting theorem (65) shows that the unit incident electron beam energy flux is divided between the outgoing wave and the outgoing electron beam. Clearly, we would like the outgoing wave energy flux as large as possible. Hence, we choose $2kF^2/d$ as the figure of merit; this quantity clearly ranges between 0 and 1.

Characterization of the orbits and explicit form of the constraint (67). When we solve (59) for $I(\psi)$, assuming $|F|$ small, there are two distinct classes of trajectories. One of these classes then divides into two distinct cases, so that there are finally three separate cases.

Case I. If

$$c - k + 2I_0 \neq 0 \tag{72}$$

then

$$I = I_0 + F\sqrt{2I_0}\cos\psi/(c - k + 2I_0) + O(F^2). \tag{73}$$

The condition that $I(\psi)$ enclose a domain of unit area, (67), is simply

$$I_0 = 1/2 + O(F^2). \tag{74}$$

The second class of trajectories is more complex and assumes

$$c - k + 2I_0 = 0 \tag{75}$$

in which case

$$I = I_0 + I_1 + \cdots \tag{76}$$

where

$$I_1^2(\psi) - F\sqrt{2I_0}\cos\psi = \text{constant}. \tag{77}$$

Thus, we may introduce a constant $\lambda > -1$ such that

$$I_1(\psi) = \pm\sqrt{F}(2I_0)^{1/4}\sqrt{\lambda + \cos\psi}. \tag{78}$$

This group of orbits also divides into two classes.

Case II. If

$$\lambda > 1 \tag{79}$$

then all values of ψ are admissible in (78) and the area condition becomes

$$2I_0 \pm \frac{1}{\pi}\sqrt{F}(2I_0)^{1/4}\int_0^{2\pi}\sqrt{\lambda + \cos\psi}\,\mathrm{d}\psi = 1 + O(F). \tag{80}$$

Case III. If $-1 < \lambda < 1$, so that

$$\lambda = -\cos\psi_M, \qquad 0 < \psi_M < \pi \tag{81}$$

then

$$-\psi_M < \psi < \psi_M \tag{82}$$

and the function $I(\psi)$ is double valued in ψ, where one takes the plus sign in (78) on one branch and then the minus sign on the other branch. In this case, the domain is roughly lune shaped and the area condition has an entirely different structure, so that

$$1 = \frac{2}{\pi}\sqrt{F}(2I_0)^{1/4}\int_{-\psi_M}^{\psi_M}\sqrt{\lambda + \cos\psi}\,\mathrm{d}\psi. \tag{83}$$

This condition can only be satisfied for large values of I_0, specifically $I_0 = O(F^{-2})$. It will become clear shortly that no acceptable solutions for this case exist. We have now completed the characterization of the orbits and determined the explicit form of the constraint (67).

The constraints (65) *and* (66). We must now examine the other two constraints (65) and (66). These constraints involve the additional parameters Γ_2 and d. In addition k, which does appear in (73) and implicitly in (83) through (75), occurs in (65), (66) in a more sensitive manner.

Case I. In this case, (72)–(74) apply and in leading order in F

$$2I = 2I_0 + O(F)\cos\psi + O(F^2) \tag{84}$$

or

$$2I = 1 + O(F)\cos\psi + O(F^2) \tag{85}$$

and thus (65) becomes

$$2kF^2/d = O(F^2). \tag{86}$$

This class of solutions has very low efficiency of transfer of energy flux from electrons to the wave.

We next turn to the two cases when $c - k + 2I_0 = 0$, given by (75)–(83). We find here

$$\frac{d\psi}{d\zeta} = \mp 2\sqrt{F}(2I_0)^{1/4}\sqrt{\lambda + \cos\psi} + \cdots. \tag{87}$$

Case II. We start with the case $\lambda > 1$, for which $I_1(\psi)$ is given by (78) and the area relation is (80). Now (65) becomes

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\psi}{\sqrt{\lambda + \cos\psi}}\right)kF^2 = \pm\sqrt{F}d\left(\frac{1}{2\pi}\int_0^{2\pi}d\psi\sqrt{\lambda + \cos\psi}\,\frac{1}{2\pi}\int_0^{2\pi}\frac{d\psi}{\sqrt{\lambda + \cos\psi}} - 1\right). \tag{88}$$

The Schwarz inequality shows that the right-hand side is positive only for plus (+) sign. Further, d/k must be small of order $F^{3/2}$. With this scaling F is determined. Finally, with $d/k = O(F^{3/2})$ (66) reduces to

$$k = \sqrt{\Gamma_2} + O(F). \tag{89}$$

Thus, acceptable final states exist if $F = O(d/k)^{2/3}$, and $F(k/d)^{2/3}$ and λ are chosen to satisfy (88). It is easy to verify that as $\lambda \to 1$

$$kF^{3/2}/d = \frac{2\sqrt{2}}{\pi}, \tag{90}$$

while for $\lambda \to \infty$

$$kF^{3/2}/d \to 0. \tag{91}$$

Clearly, solutions in λ exist for all intermediate values of $kF^{3/2}/d$. In this analysis only the upper sign is allowed in (78) and (80). We may easily estimate the energy flux conversion efficiency if we note

$$kF^2/d = O(F^{1/2}). \tag{92}$$

More precisely, as $\lambda \to 1$ we have

$$2kF^2/d = \frac{4\sqrt{2}}{\pi}\sqrt{F} + O(F), \tag{93}$$

and at $F = 0.1$ the efficiency would be 0.57. While such values of F may exceed the range of validity of the expansion, the result suggests that reasonable efficiencies are likely.

Case III. In this case, $c - k + 2I_0 = 0$, $-1 < \lambda < 1$, and the area condition is (83). Since $I_0 = O(F^{-2})$, it is easy to see that in the remaining constraint (65) the ratio of the two integrals is $O(F^{-2})$ and no relevant solutions of (65) are possible. Thus, we are finally led to reject this case as unphysical.

In summary, of the three cases one has very small energy flux conversion and one is unphysical, but in the second case reasonable energy flux conversions seem possible.

4. Conclusions

We have examined a simple model of the electron interaction with the high frequency field within a gyrotron cavity with the goal of exploring properties of solutions. We have shown first that for a limited range in axial extent solutions always exist. For one particular class of interactions, we also showed that solutions exist for any axial extent. In this case, the smoothness of solutions is the same as the assumed smoothness of the initial beam profile. We obtained directly from the equation the Poynting theorem for the constancy of energy flux.

The electron motion was shown to be a Hamiltonian dynamical system, from which follows constancy of the area in phase space confined by the beam.

We then turned to the initial and final states of a gyrotron. It is easy to obtain the standard starting conditions for the electromagnetic wave together with corrections to the beam structure of comparable magnitude. We consider possible final states in a cavity of uniform radius and a purely outgoing wave. We present an electron distribution function structure consistent with the equations of motion and dynamically accessible from the initial state. We study the final state by perturbation expansion in the wave amplitude. It appears that multiple solutions are possible. We have also identified a class of states with reasonable energy flux conversion from the beam to the wave.

References

[1] Dumbrajs O, Meyer-Spasche R and Reinfelds A 1998 Analysis of electron trajectories in a gyrotron resonator *IEEE Trans. Plasma Sci.* **26** 846–53

[2] Götz M 2006 Zur mathematischen Modellierung und Numerik eines Gyrotron Resonators *Dissertation in Mathematik* TU München 124 pp (http://mediatum2.ub.tum.de)

[3] Goldstein H, Poole Ch and Safko J 2000 *Classical Mechanics* 3rd edn (Reading, MA: Addison-Wesley)

[4] Kartikeyan M V, Borie E and Thumm M K A 2004 *Gyrotrons* (Berlin: Springer)

[5] Nusinovich G S 2004 *Introduction to the Physics of Gyrotrons* (Baltimore, MD: Johns Hopkins University Press)